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Symbolic Computation via Gröbner Basis

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ABSTRACT
The purpose of this paper is to find the orthogonal projection of a rational parametric curve onto a rational parametric surface in 3-space. We show that the orthogonal projection problem can be reduced to the problem of finding elimination ideals via Gröbnerbasis. We provide a computational algorithm to find the orthogonal projection, and include a few illustrative examples. The presented method is effective and potentially useful for many applications related to the design of surfaces and other industrial and research fields.
***Keywords*–** Orthogonal Projection, Parametrization, Polynomial,Surface

I. INTRODUCTION

A rational parametrized curve ***C*** is defined as the image of:

$$F:\mathbb{R}\to\mathbb{R}^3, t\to F(t)=\left(\frac{f_1(t)}{f_0(t)},\frac{f_2(t)}{f_0(t)},\frac{f_3(t)}{f_0(t)}\right),$$

where$f_i(t)\in\mathbb{R}[t]$are polynomials for $i=0,1,2,3$.
A rational parametrized surface ***S*** is defined as the image of:

$$G:\mathbb{R}^2\to\mathbb{R}^3,(u,v)\to G(u,v)=\left(\frac{g_1(u,v)}{g_0(u,v)},\frac{g_2(u,v)}{g_0(u,v)},\frac{g_3(u,v)}{g_0(u,v)}\right),$$

where$g_i(u,v)\in\mathbb{R}[u,v]$are polynomials for $i=0,1,2,3$.
The orthogonal projection of ***C*** onto ***S*** is defined to be the set ***H*** such that:

$$H=\{(u,v,t)\in\mathbb{R}^3 | (G(u,v)-F(t))\times N(u,v)=0\},$$

Where***N(u,v)*** is the normal vector of ***G(u,v)*** of ***S*** at ***(u,v).***
The orthogonal projection problem is to find the solution of the equation:

$$(G(u,v)-F(t))\times N(u,v)=0$$

The problem of orthogonal projection of a space curve onto a surface has attracted interest due to its importance in geometric modeling, surface design, computer graphics, cutting, patching and welding of free-form surfaces, as well as other scientific research. The orthogonal projection of a curve onto two surfaces to be blended provides not only a trimming curve design tool, but it is used to construct smooth natural maps between trimming curves on different surfaces [1].
In 1996,Pegna and Wolter[1] formulated the orthogonal projection in terms of a system of differential equations, which was solved numerically. They obtained a sequence of points along the resulting curve and used the interpolation method to obtain the approximate projection curve.
Since then different methods were presented to improve the solution. Some authors presented methods for finding a good initial point on a surface. Some authors proposed algorithms for special kinds of curves and surfaces such as NURBS. Others improved algorithms using surface properties for obtaining a better accuracy and faster convergence.
There are some drawbacks in all these algorithms. They provide approximate solutions only. Sometimes, the accuracy of the approximate curve cannot be controlled. Perhaps the main drawback is the fact that these solutions do not provide the full set of possible solutions.
The most recent article concerning this topic is byGan and Zhou [2], where the authors present three algorithms to compute orthogonal projections of rational curves onto rational parametrized surfaces by symbolic methods. To our knowledge, this is the first paper that approaches this problem from a mathematical standpoint. Though the approach we use in this paper is similar, this work is completely independent and was done before they posted their article online.
In this paper, we will show that the orthogonal projection problem can be reduced to the problem of finding elimination ideals via Gröbnerbasis. We will provide a computational algorithm to find orthogonal projection, and include a few illustrative examples.
This paper is organized in three sections: Section 2 is devoted to introducing preliminary definitions and theorems; Section 3 focuses on our main results, where we provide a computational algorithm for the

orthogonal projection problem and some illustrative examples; Conclusionbrings us a few potential interesting problems.

II. ELIMINATION THEORY

Gröbner bases were proposed by Buchberger[3] for efficientcomputation in polynomial rings. This method requires an orderingof the monomials in the polynomial ring, and the algorithm gives abasis of the ideal generated by the polynomials. We willdescribe the method starting with the orderings of the monomials.

We know that the single variable polynomial $f(x) = x^5 - 3x^2 + 1$ inring $k[x]$ is written in decreasing order of the degree in thevariable x. For multivariable monomials$x^{\propto} = {x_1}^{\propto_1} \dots {x_n}^{\propto_n}$ and $x^{\beta} = {x_1}^{\beta_1} \dots {x_n}^{\beta_n}$ in the polynomial ring $k[x_1, \dots, x_n]$, with $\propto = (\propto_1, \dots, \propto_n)$ and $\beta = (\beta_1, \dots, \beta_n)$ in $\mathbb{Z}^n_{\geq 0}$, where $\mathbb{Z}_{\geq 0}$ is the set of non-negative integers. Any ordering on $\mathbb{Z}^n_{\geq 0}$ will give an ordering on the monomials. If $\propto > \beta$, we will say $x^{\propto} > x^{\beta}$.

Definition 2.1. We say $\propto > \beta$with *Lexicographic order*, or *lexorder*,if in the vector difference$\propto - \beta \in \mathbb{Z}^n$, the left-most nonzero entry is positive. We will write $x^{\propto} > x^{\beta}$if $\propto > \beta$.

Definition 2.2. (Definition 7, Page 58, [4]). Let$f = \sum_{\propto} a_{\propto} x^{\propto}$ be a nonzeropolynomial in $k[x_1, \dots, x_n]$ and let $>$be a monomialorder.

The *multidegree* of $f$is:

$$multideg(f) = \max(\propto \in \mathbb{Z}^n_{\geq 0} : a_{\propto} \neq 0).$$

The *leading coefficient* of $f$is:

$$LC(f) = a_{multideg\ (f)} \in k.$$

The *leading monomial*of $f$is:

$$LM(f) = x^{multideg\ (f)}\ with\ coefficient\ 1.$$

The *leading term*of $f$is:

$$LT(f) = LC(f) \cdot LM(f).$$

Example 2.3. This example illustrates lex order and the monomial order corresponding todifferent variable order.

1. $(1,1,3) > (0,3,4)$
since$(1,1,3) - (0,3,4) = (1, -2, -1)$.
2. There are many lex orders corresponding to how the variables are ordered. Let

$$f = xy^2z^3 + 4x^2yz^4 \in k[x, y, z].$$

a) If $x > y > z$, we can order the terms of f in decreasing order:
$f = 4x^2yz^4 + xy^2z^3$, since $(2,1,4) > (1,2,3)$.
$multideg(f) = (2,1,4)$, $LC(f) = 4$,
$LM(f) = x^2yz^4$, $LT(f) = 4x^2yz^4$.

b)] If $y > x > z$, we can order the terms of f in decreasing order:
$f = xy^2z^3 + 4x^2yz^4$, since $(2,1,3) > (1,2,4)$.
$multideg(f) = (2,1,3)$, $LC(f) = 1$,
$LM(f) = xy^2z^3$, $LT(f) = xy^2z^3$.

Definition 2.4. For a fixed monomial order, a finite subset $G = \{g_1, \dots, g_s\}$ of an ideal I is said to be a Gröbnerbasis if

$$\langle LT(g_1), \dots, LT(g_s) \rangle = \langle LT(I) \rangle.$$

where$\langle LT(I) \rangle$ is the ideal generated by theleading terms of all the elements in I.

The process for calculating this basis is known as Buchberger'sAlgorithm. For details, see Cox, Little and O'Shea[4] (Page 108-110) andClark[5] (Page 84). This algorithm, while relatively easy tounderstand, is not usually conducive to hand computation, so aComputer Algebra System is the means by which a Gröbner Basis ismost often calculated.

Example 2.5. This example is shownin [4] (Page 75).Assume Lex order with$x > y$. Let $f, g \in R = \mathbb{C}[x, y]$and $\{f, g\} = \{x^3 - 2xy, x^2y - 2y^2 + x\}$. We claim $\{f, g\}$ is not a Gröbnerbasis for $I = \langle f, g \rangle$. Since

$$\begin{aligned} xg - yf &= x(x^2y - 2y^2 + x) - y(x^3 - 2xy) \\ &= x^3y - 2xy^2 + x^2 - x^3y - 2xy^2 = x^2, \end{aligned}$$

Then$x^2 \in \langle LT(I) \rangle$, but $x^2 \notin \langle LT(f), LT(g) \rangle = \langle x^3, x^2y \rangle$.

We can find the Gröbner basis for $I$by using the followingcommand in Mathematica:
$GroebnerBasis[\{x^3 - 2xy, x^2y - 2y^2 + x\}, \{x, y\}]$, and the Gröbner basis for I is $\{-y^4 + 2y^7 - 4y^8 +$ *2y9, x−2y3+4y5−8y6+4y7.*

Theorem 2.6. (Corollary 6, Page 76; Theorem 2, Page 89; [4]). Fix a monomial order. Then every nonzero ideal$I \subset k[x_1, \dots, x_n]$ has a Gröbner basis. Any Gröbner Basis of anideal I is a basis for I. Furthermore,theGröbner basis canbe constructed by Buchberger's Algorithm.

In commutative algebra and algebraic geometry, elimination theory is the classical name for algorithmic approaches to eliminating some variables between polynomials of several variables. One of the important application in elimination theory is to find the implicit equations of a given set of parametric expressions.

Definition 2.7. Let $I = \langle f_1, \dots, f_s \rangle \subseteq \mathbb{K}[x_1, \dots, x_n]$. The ℓ-the*elimination ideal*I_ℓ is the ideal of $\mathbb{K}[x_{\ell+1}, \dots, x_n]$ defined by

$$I_\ell = I \cap \mathbb{K}[x_{\ell+1}, \dots, x_n].$$

Theorem 2.8. (Elimination Theorem) Let $I \subseteq \mathbb{K}[x_1, \dots, x_n]$ be an ideal, and let$G$be a Gröbner basis of I with respect to lex order where$x_1 > x_2 > \cdots > x_n$. Then,for every$0 \le \ell \le n$,the set $G_\ell = G \cap \mathbb{K}[x_{\ell+1}, \dots, x_n]$is a Gröbner basis of the ℓ-th elimination idealI_ℓ.

III. ORTHOGONAL PROJECTION OF CURVES ONTO SURFACES

In this section, we will see the orthogonal projection of a rational space curve to a rational surface is an application of elimination theory, i.e., finding the elimination ideal.

A rational parametrized curve ***C*** is defined as the image of:

$$F: \mathbb{R} \to \mathbb{R}^3, t \to F(t) = \left(\frac{f_1(t)}{f_0(t)}, \frac{f_2(t)}{f_0(t)}, \frac{f_3(t)}{f_0(t)}\right),$$

where$f_i(t) \in \mathbb{R}[t]$are polynomials for $i = 0,1,2,3$.

A rational parametrized surface ***S*** is defined as the image of:

$$G: \mathbb{R}^2 \to \mathbb{R}^3, (u,v) \to G(u,v) = \left(\frac{g_1(u,v)}{g_0(u,v)}, \frac{g_2(u,v)}{g_0(u,v)}, \frac{g_3(u,v)}{g_0(u,v)}\right),$$

where$g_i(u,v) \in \mathbb{R}[u,v]$are polynomials for $i = 0,1,2,3$.

The orthogonal projection of ***C*** onto ***S*** is defined to be the set ***H*** such that:

$$H = \{(u,v,t) \in \mathbb{R}^3 \mid (G(u,v) - F(t)) \times N(u,v) = 0\},$$

where$N(u,v)$is the normal vector of $G(u,v)$of ***S*** at ***(u,v).***

In particular, we can focus our attention to the two dimensional space and study the set $E = \{(u,v) \in \mathbb{R}^2 \mid (u,v,t) \in H\}$, since this set provides the same information in a lower dimension.

Since $N(u,v) = G_u(u,v) \times G_v(u,v)$, we must have that

$$\begin{cases} (G(u,v) - F(t)) \cdot G_u(u,v) = 0 \\ (G(u,v) - F(t)) \cdot G_v(u,v) = 0 \end{cases}$$

which can be further simplified as

$$\begin{cases} \dfrac{\sum_{i=1}^{3}(g_i f_0 - f_i g_0)(g_{iu} g_0 - g_i g_{0u})}{{g_0}^3 f_0} = 0 \\ \dfrac{\sum_{i=1}^{3}(g_i f_0 - f_i g_0)(g_{iv} g_0 - g_i g_{0v})}{{g_0}^3 f_0} = 0 \end{cases}$$

Thus we have the following result concerning the orthogonal projection.

Corollary 3.1. Given a rational parametrized curve***C***:

$$F: \mathbb{R} \to \mathbb{R}^3, t \to F(t) = \left(\frac{f_1(t)}{f_0(t)}, \frac{f_2(t)}{f_0(t)}, \frac{f_3(t)}{f_0(t)}\right),$$

and a rational parametrized surface ***S***:

$$G: \mathbb{R}^2 \to \mathbb{R}^3, (u,v) \to G(u,v) = \left(\frac{g_1(u,v)}{g_0(u,v)}, \frac{g_2(u,v)}{g_0(u,v)}, \frac{g_3(u,v)}{g_0(u,v)}\right),$$

Where$f_i(t) \in \mathbb{R}[t]$, $g_i(u,v) \in \mathbb{R}[u,v]$are polynomials for$i = 0,1,2,3$. Fix a monomial order with$s > t > u > v$, and let Gbe a Gröbnerbasis of the ideal generated by the following three polynomials:

$$\sum_{i=1}^{3}(g_i f_0 - f_i g_0)(g_{iu} g_0 - g_i g_{0u}),$$
$$\sum_{i=1}^{3}(g_i f_0 - f_i g_0)(g_{iv} g_0 - g_i g_{0v}),$$
$$s g_0 f_0 - 1.$$

Then$H$, the orthogonal projection of***C*** onto ***S***, is the variety of the first elimination ideal,
$H = \mathbb{V}(G \cap \mathbb{R}[t,u,v])$, moreover,
$E = \mathbb{V}(G \cap \mathbb{R}[u,v])$.

Proof. The result follows directly from the elimination theorem. □

Algorithm.
Input: A parametric curve and a surface given as:

$$F = (f_0(t), f_1(t), f_2(t), f_3(t)),$$
$$G = (g_0(u,v), g_1(u,v), g_2(u,v), g_3(u,v)).$$

Output: Orthogonal projection$E = \mathbb{V}(G \cap \mathbb{R}[u,v])$.
Procedure:
Step 1. Calculate partial derivatives $g_{iu}(u,v)$ and $g_{iv}(u,v)$for $i = 0,1,2,3$.
Step 2. Construct polynomials:
$h_1 = \sum_{i=1}^{3}(g_i f_0 - f_i g_0)(g_{iu} g_0 - g_i g_{0u})$,
$h_2 = \sum_{i=1}^{3}(g_i f_0 - f_i g_0)(g_{iv} g_0 - g_i g_{0v})$,
$h_3 = s g_0 f_0 - 1$.
Step 3. Compute the Gröbnerbasis $G$of the ideal generated by h_1, h_2, h_3 with respect to lexicographic order $s > t > u > v$.
Step 4. Identify the second elimination ideal:

$$E = \mathbb{V}(G \cap \mathbb{R}[u,v])$$

The following examples illustrate our algorithm.

Example 3.2. Let the space curve ***C*** and the surface ***S*** be given as:
$F(t) = \left(\frac{t}{t+1}, \frac{t^2}{t+1}, \frac{t^3}{t+1}\right)$,
$G(u,v) = (2u, 2v, u^2 + v^2 - 5)$.
Then by the above result, we obtain
$G \cap \mathbb{R}[u,v] = 9u^3 - 6u^5 + u^7 + 3uv + 9u^2 v - u^3 v - 6u^4 v + u^6 v - 6u^3 v^2 + 2u^5 v^2 + 3v^3 - uv^3 - 7u^2 v^3 + 2u^4 v^3 + u^3 v^4 - v^5 + u^2 v^5$.

$$E = \mathbb{V}(G \cap \mathbb{R}[u, v]).$$

Below, we provide the three dimension figures of the space curve and the surface, together with their two dimensional orthogonal projection image (Fig.1).

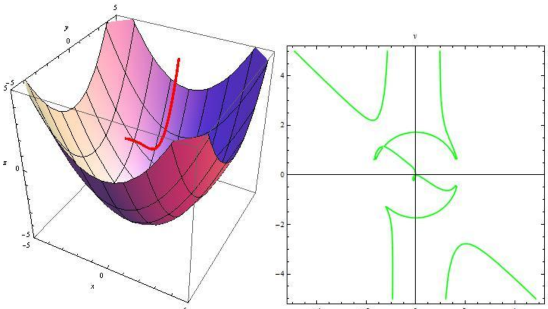

Figure 1. Curve C in red and surface S (left); Orthogonal projection (right)

Example 3.3. Let the space curve ***C*** and the surface ***S*** be given as:

$$F(t) = \left(\frac{t^3}{t+1}, \frac{t^2}{t+1}, \frac{t}{t+1}\right),$$

$$G(u, v) = \left(\frac{uv}{u+v}, \frac{uv^2}{u+v}, \frac{u^2}{u+v}\right).$$

Then by the above result, we obtain

$$G \cap \mathbb{R}[u, v] = -u^{12} + u^9 v - 2\,u^{10} v - 4\,u^{11} v + 2\,u^{12} v + 5\,u^8 v^2 - 8\,u^9 v^2 - 9\,u^{10} v^2 + 7\,u^{11} v^2 + 10\,u^7 v^3 - 12\,u^8 v^3 - 11\,u^9 v^3 + 4\,u^{10} v^3 + u^3 v^{14} + 2\,u^{11} v^3 + 10\,u^6 v^4 - 8\,u^7 v^4 - 2\,u^8 v^4 + 3\,u^4 v^{14} - 14\,u^9 v^4 + 7\,u^{10} v^4 + 5\,u^5 v^5 - 2\,u^6 v^5 + u^3 v^{15} + 14\,u^7 v^5 - 30\,u^8 v^5 + 7\,u^9 v^5 + u^4 v^6 + 26\,u^6 v^6 - 26\,u^7 v^6 - 11\,u^8 v^6 - 4\,u^9 v^6 + 26\,u^5 v^7 - 8\,u^6 v^7 - 25\,u^7 v^7 - 14\,u^8 v^7 + 14\,u^4 v^8 + 2\,u^5 v^8 - 3\,u^6 v^8 - 18\,u^7 v^8 - 4\,u^8 v^8 + 3\,u^3 v^9 + 27\,u^5 v^9 - 9\,u^6 v^9 - 10\,u^7 v^9 - u^3 v^{10} + 30\,u^4 v^{10} + 3\,u^5 v^{10} - 9\,u^6 v^{10} + 11\,u^3 v^{11} + u^4 v^{11} + 2\,u^5 v^{11} + 2\,u^6 v^{11} - u^2 v^{12} - 3\,u^3 v^{12} + 11\,u^4 v^{12} + 3\,u^5 v^{12} + 6\,u^3 v^{13} + 2\,u^5 v^{13}.$$

$$E = \mathbb{V}(G \cap \mathbb{R}[u, v]).$$

Below, we provide the three dimension figures of the space curve and the surface, together with their two dimensional orthogonal projection image (Fig.2).

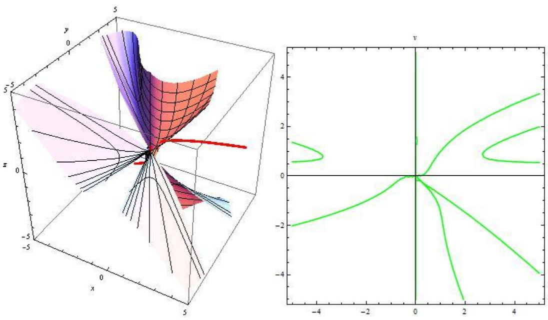

Figure 2.Curve C in red and surface S (left); Orthogonal projection (right)

Example 3.4. Let the space curve ***C*** and the surface ***S*** be given as:

$$F(t) = \left(t, 1, \frac{1}{t^2}\right),$$

$$G(u, v) = \left(\frac{8u}{(1+u^2)(1+v^2)}, \frac{8v}{(1+u^2)(1+v^2)}, \frac{10(u^2+v^2)}{(1+u^2)(1+v^2)}\right).$$

The resulting polynomial $G \cap \mathbb{R}[u, v]$ has 128 terms; the highest exponents of the variables u and v are 18 and 17 respectively. Nevertheless, it is still an implicit equation of a curve on a u,v-plane, which we can see in Fig.3.

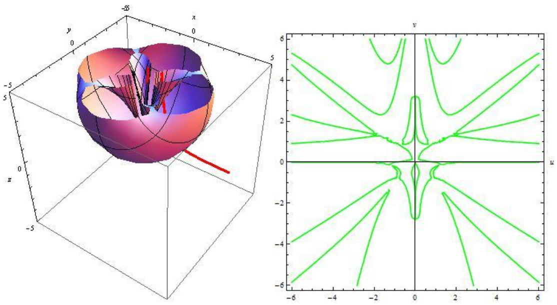

Figure 3.Curve C in red and surface S (left); Orthogonal projection (right)

CONCLUSION

In this paper, we solve the orthogonal projection problem via Gröbnerbasis method. This paper can be interesting and, hopefully, helpful to engineers designing complex surfaces in various industries and in science.

The main contribution of this paper is the idea of receiving an exact polynomial equation for the orthogonal projection of a parametric rational curve onto a parametric rational surface.

There are some questions that might be interesting for future research:

Question 1. Having a system of polynomial equations, how we can determine the best elimination order to increase the computation speed?

Question 2. Is there any other method that can simplify and expedite the computation?

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